11-2020

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The Revealed Preference Theory of Stable Matchings with One-Sided Preferences*

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May 3, 2019

Abstract

This note studies the testable implications of the theory of stable matchings in two-sided matching markets with one-sided preferences. Our main result connects the revealed preference analysis to the well-known lattice structure of the set of stable matchings, and tests the rationalizability of a data set by analyzing the joins and meets of matchings.

JEL Classification: C78, D80

Keywords: revealed preference; stable matchings; one-sided preferences; lattice structure

*We are indebted to the advisory editor and anonymous referees for detailed comments and suggestions that substantially improved the paper. We would like to thank David Ahn, Fuhito Kojima, John Quah, Xiang Sun, Satoru Takahashi, and Yanqi Zhu for helpful discussions. Li gratefully acknowledges the hospitality of the Department of Economics at the Northwestern University, where part of this was written. All remaining errors are our own.

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1 Introduction

Revealed preference theory is based on the idea that the preference of the decision maker (DM) is revealed from her choice behavior (see Chambers and Echenique (2016) for a comprehensive survey). Alternative x is revealed to be preferred to alternative y if and only if x is chosen when y is also available. Note that, to draw inferences from the choice behavior, one need not only observe the alternative that is chosen, but also which other alternatives are available to the DM.[1]

The latter requirement that the observer observes what alternatives are available to the DM presents some challenges in the matching markets. For a simple example, in the school choice setting, if Ann is matched with School A and not with School B, it may be that she prefers School A to School B, but it may also be that School B admits other students who have higher priorities than Ann at School B.

In this note, we study the testable implications of the theory of stable matchings in two-sided matching markets with one-sided preferences. Throughout this note, we present our results using the school choice model in which students are matched with schools. It is assumed that the schools' priority orderings are known. A data set of matchings is said to be *rationalizable* if there exists a profile of the students' preferences such that every matching in the data set is stable given the students' preferences and the schools' priority orderings.

We provide two approaches to test the rationalizability of a data set. As we assume that the schools' priority orderings are known, a natural way to proceed is to first identify the alternatives available to each student, and then reveal the students' preferences. For example, suppose that Ann is matched with School A and Bob is matched with School B. If we know that Ann has higher priority than Bob at School B, then we can conclude that School B is available to Ann. Since Ann is matched with School A, we can further conclude that Ann prefers School A to School B. Indeed, the rationalizability of a data set can be characterized via an acyclicity condition, that is, the revealed preferences of each student are not cyclical.

Our main result provides an indirect approach to test the rationalizability of a data set. This approach connects the revealed preference analysis to the well-known lattice structure of the set of stable matchings, where the latter says that the set of stable matchings forms a

[1] Hu et al. (2018) study the (coarse) revealed preference theory in settings in which the observer does not observe the DM's exact choice.

distributive lattice (see, for example, Knuth (1976), Blair (1984), and Roth and Sotomayor (1990)).

The exact conditions for rationalizability of a data set will be made clear in the formal analysis. Here, we provide some discussion on how to apply the lattice structure result to conduct the revealed preference test. For any two mappings μ and μ', we define their *join* (with respect to the schools) $\lambda = \mu \vee \mu'$ such that λ maps each school s to its preferred set of students between $\mu(s)$ and $\mu'(s)$. Analogously, we define their *meet* (with respect to the schools) $\nu = \mu \wedge \mu'$ such that ν maps each school s to its less preferred set of students between $\mu(s)$ and $\mu'(s)$. Suppose that a data set D is rationalizable. By definition, there exists a profile of the students' preferences such that every matching in D is stable given the students' preferences and the schools' priority orderings. It follows from the lattice structure result that the join and meet of any two matchings in D are well-defined matchings. Thus, if the join (or the meet) is not a well-defined matching, for example, if a student is mapped to two schools in the join (or the meet), then the data set is not rationalizable. Example 2 in Section 3.2 applies this logic to show that a given data set is not rationalizable by constructing the join and meet of two matchings and verifying that they are not well-defined matchings.

The logic above can be substantially generalized by iteratively constructing the join and meet of mappings. If at any stage, either the join or the meet of two mappings is not a well-defined matching, then we can refute the rationalizability of the data set. Utilizing the lattice structure result, we establish that a necessary condition for the rationalizability of a data set is that all the joins and meets that are iteratively constructed in this way are well-defined matchings. However, this is not sufficient, as illustrated by Example 3 and Example 4 in section 3.2.

To deal with the issues associated with these two examples, we also connect our revealed preference analysis to (1) the celebrated Lone Wolf Theorem (see, for example, Roth and Sotomayor (1990, Theorem 5.12)) that states that the set of students matched and school seats filled is the same for all stable matchings; and (2) Theorem 5.27 in Roth and Sotomayor (1990) that states that the set of each school's matches in stable matchings is responsively ordered.[2]

Our main result, Theorem 1, shows that a data set D is rationalizable if and only if

[2]We say that a school's matches in a set of matchings are responsively ordered if for any two matchings in the set μ and μ', this school prefers its match in μ to its match in μ' implies that this school prefers every student it matches with at μ to every student it matches with at μ' but not at μ.

1. every mapping that is iteratively constructed by taking the joins and meets is a well-defined matching;
2. the set of students matched and school seats filled is the same for all matchings in D;
3. the set of each school's matches in D is responsively ordered.

These conditions are tight. We have a series of examples illustrating that none of the three conditions can be dropped. For each condition, we have an example of a data set in which the other two conditions are satisfied and the data set is not rationalizable.

Our proof is constructive. If a data set is rationalizable, we explicitly construct a class of the students' preferences such that every matching in the data set is stable given any preferences of the students in this class and the schools' priority orderings.

In the special case of one-to-one matching markets, the third condition that the set of each school s's matches in D is responsively ordered is vacuous. Thus, the first two conditions are necessary and sufficient for the rationalizability of a data set in one-to-one matching markets.

Echenique (2008) studies the testable implications of two-sided matching theory when the preferences of agents on both sides of the market are unknown. In contrast, we assume that the schools' priority orderings are known. This additional structure enables us to derive sharper results by connecting the revealed preference analysis to the lattice structure of the set of stable matchings. In particular, one can work out the join and meet of two mappings only when the preferences of agents on one side of the market are known. More recently, Echenique et al. (2013) investigate the revealed preference theory of aggregate matchings, which differ from the individual level matchings studied in the current paper.

Haeringer and Iehlé (forthcoming) also study two-sided matching markets with one-sided preferences. By considering only the preferences of agents on one side of the market, they show how one can identify impossible matches, i.e., pairs of agents that can never be matched together. Identifying impossible matches is extremely valuable, because they can be used to narrow down the set of stable matchings that may emerge. This note complements their paper and asks, for a given data set of matchings, whether there exists a profile of the students' preferences such that every matching in the data set is stable given the students' preferences and the schools' priority orderings. To relate the results in Haeringer and Iehlé (forthcoming) to our analysis, if any of the matchings in the data set contains an impossible match, the data set is not rationalizable. But if none of the matchings in the data set contains an impossible match, their results are silent about the rationalizability of the data set.

2 Model

The school choice model. Throughout the paper, we shall present our results using the school choice model in which students are matched with schools. There is a finite set I of students to be matched with a finite set S of schools. We denote by i a generic student, denote by s a generic school, and denote by a a generic agent (student or school). The capacity of school s is q_s, i.e., school s can admit at most q_s students.

Each student $i \in I$ has a strict preference $\succ_i$ over $S \cup \{i\}$, where i stands for the outside option of remaining unmatched. The relation $s \succ_i s'$ means that student i prefers to be matched with school s rather than being matched with school s', and the relation $s \succ_i i$ means that student i prefers to be matched with school s rather than remaining unmatched. Each student i's preference $\succ_i$ over $S \cup \{i\}$ is her private information.

Each school $s \in S$ is endowed with a strict priority ordering $\succ_s$ over $I \cup \{s\}$, where s stands for the outside option of being unmatched. The relation $i \succ_s i'$ means that student i has higher priority than student i' at school s, and the relation $i \succ_s s$ means that school s prefers to be matched with student i rather than remaining unmatched. Note that even though we have specified each school's priority ordering over individual students, each school with capacity larger than one must be able to compare sets of students. We assume that each school's priority ordering over sets of students is responsive.[3] A priority ordering over sets of students is responsive (to the priority ordering over individual students) if for any two subsets A and A' of $I \cup \{s\}$ such that they differ by only one element, i.e., $A' = A \cup \{i'\} \setminus \{i\}$ (resp. $A' = A \cup \{i'\}$), A' has a higher priority than A at school s if and only if $i' \succ_s i$ (resp. $i' \succ_s s$). To economize on notations, we shall also use $\succ_s$ to denote the responsive priority ordering over sets of students. This should not cause any confusion. The schools' priority orderings $\{\succ_s\}_{s \in S}$ are assumed to be known.

A *school choice market* is denoted by $\Gamma = (I, S, \{\succ_i\}_{i \in I}, \{q_s, \succ_s\}_{s \in S})$. For a given relation $\succ_a$, we write $\succcurlyeq_a$ for the weak relation associated with it, i.e., $a'' \succcurlyeq_a a' \iff a'' \succ_a a'$ or $a'' = a'$.

Matchings and stable matchings. To give a formal definition of a matching, we first define for any set X an *unordered family of elements of* X to be a collection of elements, *not necessarily distinct*, in which the order is immaterial. A *matching* μ is a function from the set $I \cup S$ into the set of unordered families of elements of $I \cup S$ such that:

(1) $|\mu(i)| = 1$ for every student i and $\mu(i) = i$ if $\mu(i) \notin S$;

[3] See Roth and Sotomayor (1990, pp. 127-128) for further discussions on responsive priority orderings. We will discuss non-responsive priority orderings over sets of students in Section 4.3.

(2) $|\mu(s)| = q_s$ for every school s, and if the number of students in $\mu(s)$, say r, is less than q_s, then $\mu(s)$ contains $q_s - r$ copies of s; and

(3) $\mu(i) = s$ if and only if $i \in \mu(s)$.

In words, each student is either unmatched or assigned to a school that admits her, and each school is either matched with students up to its capacity or left with some vacancies. A matching will sometimes be represented as a set of matched pairs. For example, the matching

$$\mu = \begin{matrix} i_1, i_4, (s_1) & i_3 & i_2 \\ s_1 & s_2 & (i_2) \end{matrix}$$

has i_1 and i_4 matched with s_1 who has a capacity $q_1 = 3$, i_3 matched with s_2 who has a capacity $q_2 = 1$, and i_2 remaining unmatched. We denote by M the set of all possible matchings.

A matching μ is said to be *stable* if

(1) it is *individually rational*, i.e.,

$$\mu(i) \succcurlyeq_i i \text{ for all } i \in I, \text{ and } a \succcurlyeq_s s \text{ for all } s \in S \text{ and all } a \in \mu(s); \text{ and}$$

(2) it is *unblocked*, i.e.,

$$\text{there exists no pair } (i, s) \text{ such that } s \succ_i \mu(i) \text{ and } i \succ_s a \text{ for some } a \in \mu(s).$$

We denote by $\Sigma(\{\succ_i\}_{i\in I}, \{\succ_s\}_{s\in S})$ the set of stable matchings with respect to the students' preferences $\{\succ_i\}_{i\in I}$ and the schools' priority orderings $\{\succ_s\}_{s\in S}$. Since we assume that the schools' priority orderings are known, for notational simplicity, we write $\Sigma(\{\succ_i\}_{i\in I})$ rather than $\Sigma(\{\succ_i\}_{i\in I}, \{\succ_s\}_{s\in S})$.

The relation $\succ_a$ can be straightforwardly extended to a preference over matchings. We say that an agent prefers matching μ to matching μ' if she prefers her match in μ to her match in μ'. We abuse the notations slightly and write $\mu \succ_a \mu'$ (resp. $\mu \succcurlyeq_a \mu'$) if and only if $\mu(a) \succ_a \mu'(a)$ (resp. $\mu(a) \succcurlyeq_a \mu'(a)$). We impose the partial order $\geq_S$ on the set M, where $\mu \geq_S \mu'$ if and only if $\mu \succcurlyeq_s \mu'$ for all $s \in S$.

Rationalizability of a data set. A *data set* D is a collection of matchings $\{\mu_k\}_{k\in K}$, where K is a finite index set. The interpretation of D is that it describes K observations of the matching outcomes of a matching market.

A data set D is said to be *rationalizable* if there exists a profile of the students' preferences $\{\succ_i\}_{i\in I}$ such that

$$D \subseteq \Sigma(\{\succ_i\}_{i\in I}).$$

In words, a data set is rationalizable if there exists a profile of the students' preferences such that every matching in the data set is a stable matching given the students' preferences and the schools' priority orderings. Obviously, a data set D is rationalizable only if every matching in D is individually rational for all schools, which is a necessary condition that is immediately verifiable. To avoid redundant discussions, we assume throughout the paper that every matching in D is individually rational for all schools.

Graph-theoretic terminology. We collect some graph-theoretic terminology used in the sequel. A *directed graph* is a pair $G = (V, E)$, where V is a set of *vertices* and $E \subseteq V \times V$ is a set of *edges*. A *path* in G is a sequence $p = \langle v_0, v_1, \ldots, v_N \rangle$ such that $(v_n, v_{n+1}) \in E$ for $n = 0, 1, \ldots, N-1$. A *cycle* in G is a path $c = \langle v_0, v_1, \ldots, v_N \rangle$ with $v_0 = v_N$.

3 Results

We provide two approaches to test the rationalizability of a data set. Section 3.1 conducts the revealed preference analysis in a direct manner. That is, we first use the data set to reveal which alternatives are available to each student, and then proceed to characterize the rationalizability of a data set via an acyclicity condition. Section 3.2 connects the revealed preference analysis to the well-known lattice structure of the set of stable matchings, and tests the rationalizability of a data set by analyzing the joins and meets of matchings.

3.1 Revealed available alternatives and revealed preference

Since the schools' priority orderings are known, we can first use the data set to reveal which other alternatives are available to each student. The analysis is straightforward. For a given matching $\mu_k \in D$, school s is revealed to be available to student i if i has higher priority than some agent $a \in \mu_k(s)$ at s. More formally, for each $k \in K$ and each $i \in I$, the collection of schools that are revealed to be available to i under μ_k is

$$A(i, \mu_k) := \{s \in S : s \neq \mu_k(i) \text{ and there exists } a \in \mu_k(s) \text{ such that } i \succ_s a\}.$$

We can then proceed to reveal the students' preferences. If the data set D is rationalizable, then μ_k is a stable matching for all $k \in K$. Since the schools in $A(i, \mu_k)$ are available to student i, it must be that i prefers her match under μ_k to each school in $A(i, \mu_k)$. Since student i always has the option of remaining unmatched, it must be that i prefers her match under μ_k to remaining unmatched whenever i is matched with a school. We summarize the analysis above using the following set:

$$R(i) := \Big(\bigcup_{k\in K} \Big\{(\mu_k(i), s) : s \in A(i, \mu_k)\Big\}\Big) \bigcup \Big(\bigcup_{k\in K:\mu_k(i)\neq i} \Big\{(\mu_k(i), i)\Big\}\Big), \tag{1}$$

where

(1) $(\mu_k(i), s) \in R(i)$ means that $\mu_k(i)$ is revealed to be preferred to s by i; and

(2) $(\mu_k(i), i) \in R(i)$ means that $\mu_k(i)$ is revealed to be preferred to remaining unmatched by i.

By construction, for each $i \in I$, $(S \cup \{i\}, R(i))$ is a directed graph. Proposition 1 below characterizes the rationalizability of a data set via an acyclicity condition, that is, the revealed preferences of each student are not cyclical.

Proposition 1. *A data set D is rationalizable if and only if $(S \cup \{i\}, R(i))$ admits no cycle for all $i \in I$.*

Proof. The only if-part is trivial. We prove the if-part below. Fix an arbitrary $i \in I$. Since $(S \cup \{i\}, R(i))$ admits no cycle, by the Szpilrajn's extension theorem (Szpilrajn (1930)), there exists an extension of $\succ_i$ of $R(i)$ such that $\succ_i$ is a strict preference.

We show that $\{\succ_i\}_{i\in I}$ rationalizes D. Suppose to the contrary, D is not rationalized by $\{\succ_i\}_{i\in I}$. By definition, if the students' preferences are $\{\succ_i\}_{i\in I}$, then D contains some matching μ_k that either violates individual rationality or is blocked. If μ_k violates individual rationality, then there exists some student i such that $i \succ_i \mu_k(i)$. Since i is matched with $\mu_k(i)$ in the matching μ_k and $\mu_k(i) \neq i$, by the definition of $R(i)$, we have that $(\mu_k(i), i) \in R(i)$. This further implies that $\mu_k(i) \succ_i i$. This violates the acyclicity of $\succ_i$. If μ_k is blocked, then there exists some pair (i, s) such that $s \succ_i \mu_k(i)$ and $i \succ_s a$ for some $a \in \mu_k(s)$. By the definition of $A(i, \mu_k)$, we have that $s \in A(i, \mu_k)$. By the definition of $R(i)$, $(\mu_k(i), s) \in R(i)$. This further implies that $\mu_k(i) \succ_i s$. Again, this violates the acyclicity of $\succ_i$. □

Remark 1. *Echenique (2008) proposes a revealed preference test in a one-to-one marriage market in which the preferences of agents on both sides of the market are unknown. In this sense, our setting can be viewed as a special case of his. Indeed, Echenique (2008) also characterizes the rationalizability of data sets in graph-theoretic terms. In our setting with known priority orderings, Echenique (2008, Theorem 6.1) reduces to our Proposition 1.*

Example 1 below illustrates how to use Proposition 1 to conduct the revealed preference test.

Example 1. *Consider the matchings between students in* $I = \{i_1, i_2, i_3\}$ *and schools in* $S = \{s_1, s_2, s_3\}$*. The capacity of each school is one. The schools' priority orderings over individual students are given as follows:*

$$s_1 : i_2 \succ_{s_1} i_3 \succ_{s_1} i_1 \succ_{s_1} s_1$$

$$s_2 : i_3 \succ_{s_2} i_1 \succ_{s_2} i_2 \succ_{s_2} s_2$$

$$s_3 : i_1 \succ_{s_3} i_2 \succ_{s_3} i_3 \succ_{s_3} s_3$$

The data set consists of the following two observations $D = \{\mu, \mu'\}$*, where*

$$\mu = \begin{array}{ccc} i_1 & i_2 & i_3 \\ s_1 & s_3 & s_2 \end{array} \quad \text{and} \quad \mu' = \begin{array}{ccc} i_1 & i_2 & i_3 \\ s_3 & s_2 & s_1 \end{array}.$$

The analysis is summarized in Table 1. Since $c = \langle s_1, s_2, s_1 \rangle$ *is a cycle in the directed graph* $(S \cup \{i_3\}, R(i_3))$*, by Proposition 1, we conclude that* D *is not rationalizable.*

Table 1: Revealed Preference Analysis in Example 1.

Matchings	Students	Matches	Schools Available	Revealed Preference	
	i_1	s_1	s_3	(s_1, s_3)	(s_1, i_1)
μ	i_2	s_3	s_1	(s_3, s_1)	(s_3, i_2)
	i_3	s_2	s_1	(s_2, s_1)	(s_2, i_3)
	i_1	s_3	s_2	(s_3, s_2)	(s_3, i_1)
μ'	i_2	s_2	s_1	(s_2, s_1)	(s_2, i_2)
	i_3	s_1	s_2	(s_1, s_2)	(s_1, i_3)

3.2 The lattice structure of the set of stable matchings

In this section, we provide an indirect approach to test the rationalizability of a data set. This approach connects the revealed preference analysis to the well-known lattice structure of the set of stable matchings.

We first introduce some notations for the lattice theory. For any two mappings μ and μ', we define their *join* (with respect to $\{\succ_s\}_{s \in S}$) $\lambda = \mu \vee \mu'$ such that λ maps each school

s to its preferred set of students between $\mu(s)$ and $\mu'(s)$. Formally, for each $s \in S$,

$$\lambda(s) = \begin{cases} \mu(s) & \text{if } \mu(s) = \mu'(s); \\ \mu(s) & \text{if } \mu(s) \succ_s \mu'(s); \\ \mu'(s) & \text{if } \mu'(s) \succ_s \mu(s). \end{cases}$$

Analogously, we define their *meet* (with respect to $\{\succ_s\}_{s\in S}$) $\nu = \mu \wedge \mu'$ such that ν maps each school s to its less preferred set of students between $\mu(s)$ and $\mu'(s)$. Formally, for each $s \in S$,

$$\nu(s) = \begin{cases} \mu(s) & \text{if } \mu(s) = \mu'(s); \\ \mu(s) & \text{if } \mu'(s) \succ_s \mu(s); \\ \mu'(s) & \text{if } \mu(s) \succ_s \mu'(s). \end{cases}$$

Note that the join and the meet of two mappings may not be well defined since a school may be indifferent between two distinct sets of students. However, if μ and μ' are stable matchings, then their join and meet are well defined. This is because, under our assumptions on preferences and priority orderings, schools have strict preferences over those groups of students that they may be assigned at stable matchings (see, for example, Roth and Sotomayor (1990, Theorems 5.26)).

It is well known that the set of stable matchings forms a distributive lattice (see, for example, Knuth (1976), Blair (1984), and Roth and Sotomayor (1990, Theorem 5.31)). Formally, if μ and μ' be stable matchings, then $\lambda = \mu \vee \mu'$ and $\nu = \mu \wedge \mu'$ are well-defined matchings and are stable matchings. The lattice structure of the set of stable matchings can be used to refute the rationalizability of a data set, as illustrated by the following example.

Example 2 (Example 1 revisited)**.** *The data set and the schools' priority orderings are the same as in Example 1. We first compute the join and meet of the two matchings μ and μ' as follows:*

$$\lambda = \mu \vee \mu' = \begin{matrix} i_3 & i_3 & i_1 \\ s_1 & s_2 & s_3 \end{matrix} \quad \textit{and} \quad \nu = \mu \wedge \mu' = \begin{matrix} i_1 & i_2 & i_2 \\ s_1 & s_2 & s_3. \end{matrix}$$

Suppose that D is rationalizable. By definition, there exists a profile of the students' preferences such that both μ and μ' are stable given the students' preferences and the schools' priority orderings. By the lattice structure result, it must be that both λ and ν are stable matchings. However, neither of the two matchings λ and ν is even well defined, since λ matches student i_3 with both schools s_1 and s_2, and ν matches student i_2 with both schools s_2 and s_3. Thus, we conclude that D is not rationalizable.

More generally, let $D_0 = D$. For each integer $l \geq 0$, we define the following set D_{l+1} such that each mapping in D_{l+1} is the join of two mappings in D_l:

$$D_{l+1} := \{\mu \vee \mu' : \mu, \mu' \in D_l\}.$$

Whenever the joins are well defined, we keep constructing these sets until when $D_{\bar{l}+1} = D_{\bar{l}}$. Analogously, for each integer $l \geq 0$, we define the following set $D_{-(l+1)}$ such that each mapping in $D_{-(l+1)}$ is the meet of two mappings in D_{-l}:

$$D_{-(l+1)} := \{\mu \wedge \mu' : \mu, \mu' \in D_{-l}\}.$$

Whenever the meets are well defined, we keep constructing these sets until when $D_{-(\underline{l}+1)} = D_{-\underline{l}}$. When the joins and meets are always well defined, both $\bar{l}$ and $\underline{l}$ exist. To see this, note that we allow for the possibility that $\mu = \mu'$. Thus, $D_{l+1} \supseteq D_l$ and $D_{-(l+1)} \supseteq D_{-l}$ for all l. As such, both $\bar{l}$ and $\underline{l}$ exist because we have finitely many students and finitely many schools and thus finitely many mappings. When the joins and the meets are always well defined, we let

$$DL^+ := \bigcup_{l=0}^{\bar{l}} D_l, DL^- := \bigcup_{l=-\underline{l}}^{0} D_l, \text{ and } DL := DL^+ \cup DL^-.$$

Suppose that D is rationalizable. It follows from the lattice structure result that every mapping in DL is a well-defined matching. Thus, this is a necessary condition for rationalizability. The readers might hope that this is also sufficient for the rationalizability of a data set. This is not the case, as illustrated by Examples 3 below.

Example 3. *Consider the matchings between one student i and one school s with its capacity $q_s = 1$. The priority ordering of school s is given by $i \succ_s s$. The data set consists of the following two observations $D = \{\mu, \mu'\}$, where*

$$\mu = \begin{matrix} i \\ s \end{matrix} \qquad \text{and} \qquad \mu' = \begin{matrix} i & (s) \\ (i) & s \end{matrix}.$$

It is easy to see that $DL = D = \{\mu, \mu'\}$. Note that every mapping in DL is a well-defined matching. However, D is not rationalizable, since the stability of μ requires that $s \succ_i i$ and the stability of μ' requires that $i \succ_i s$.

In Example 3, the set of students matched and school seats filled is different in the two matchings in D. Indeed, the celebrated Lone Wolf Theorem shows that when all preferences and priority orderings over individuals are strict, the set of students matched and school seats filled is the same at every stable matching (see, for example, Roth and Sotomayor (1990, Theorem 5.12)). Thus, another necessary condition for the rationalizability of a data set D is that the set of students matched and school seats filled is the same for all matchings in D.

Even if every mapping in DL is a well-defined matching and the set of students matched and school seats filled is the same for all matchings in D, D may still not be rationalizable. We present such an example below.

Example 4. *Consider the matchings between students in $I = \{i_1, i_2, i_3, i_4\}$ and schools in $S = \{s_1, s_2\}$. The capacity of each school is two. The priority orderings of schools are given as follows:*

$$s_1: \quad i_1, i_2 \succ_{s_1} i_1, i_3 \succ_{s_1} i_1, i_4 \succ_{s_1} i_2, i_3 \succ_{s_1} i_2, i_4 \succ_{s_1} i_3, i_4 \succ_{s_1} i_1 \succ_{s_1} i_2 \succ_{s_1} i_3 \succ_{s_1} i_4 \succ_{s_1} s_1$$
$$s_2: \quad i_1, i_2 \succ_{s_2} i_1, i_3 \succ_{s_2} i_2, i_3 \succ_{s_2} i_1, i_4 \succ_{s_2} i_2, i_4 \succ_{s_2} i_3, i_4 \succ_{s_2} i_1 \succ_{s_2} i_2 \succ_{s_2} i_3 \succ_{s_2} i_4 \succ_{s_2} s_2$$

The two priority orderings differ only in the ranking between $\{i_1, i_4\}$ and $\{i_2, i_3\}$. It is easy to verify that both priority orderings are responsive. The data set consists of the following two observations $D = \{\mu, \mu'\}$, where

$$\mu = \begin{array}{cc} i_1, i_4 & i_2, i_3 \\ s_1 & s_2 \end{array} \qquad \textit{and} \qquad \mu' = \begin{array}{cc} i_2, i_3 & i_1, i_4 \\ s_1 & s_2 \end{array}.$$

It is easy to see that $DL = D = \{\mu, \mu'\}$. Note that (1) every mapping in DL is a well-defined matching; and (2) the set of students matched and school seats filled is the same in all the matchings in D. However, D is not rationalizable, since the stability of μ requires that $s_1 \succ_{i_1} s_2$ and the stability of μ' requires that $s_2 \succ_{i_1} s_1$.

We say that $\{\mu(s) : \mu \in D\}$ is responsively ordered if for any $\mu, \mu' \in D$, $\mu(s) \succ_s \mu'(s)$ implies $i \succ_s i'$ for all $i \in \mu(s)$ and $i' \in \mu'(s) - \mu(s)$.[4] That is, s prefers every student in its entering class at μ to every student who is in its entering class at μ' but not at μ. In Example 4, neither of the sets $\{\mu(s) : \mu \in D\}$ is responsively ordered. Theorem 5.27 in Roth and Sotomayor (1990) shows that in settings in which the preferences and the priority orderings over individuals are strict and the priority orderings are responsive, if μ and μ' are stable matchings and $\mu(s) \succ_s \mu'(s)$ for some school s, then $i \succ_s i'$ for all i in $\mu(s)$ and

[4] $\mu'(s) - \mu(s)$ is the difference between two unordered family of elements of $I \cup \{s\}$.

i' in $\mu'(s) - \mu(s)$. Thus, a third necessary condition for the rationalizability of a data set D is that $\{\mu(s) : \mu \in D\}$ is responsively ordered for all $s \in S$.

We have now presented three necessary conditions for the rationalizability of a data set D, namely,

1. every mapping in DL is a well-defined matching;
2. the set of students matched and school seats filled is the same for all matchings in D;
3. $\{\mu(s) : \mu \in D\}$ is responsively ordered for all $s \in S$.

As we have discussed above, these three conditions echo the lattice structure result, the Lone Wolf Theorem, and the theorem that states that the set of matches of each school under stable matchings is responsively ordered. We also illustrate via Examples 2 - 4 how these conditions can be used to refute the rationalizability of some data set. Our main result, Theorem 1, shows that these three conditions are also sufficient for the rationalizability of a data set.

Theorem 1. *A data set D is rationalizable if and only if*

(1) $DL \subseteq M$;
(2) the set of students matched and school seats filled is the same for any matching in D;
(3) $\{\mu(s) : \mu \in D\}$ is responsively ordered for all $s \in S$.

Remark 2. *Theorem 1 shows that the three conditions together are sufficient for the rationalizability of a data set. None of these conditions can be dropped for the sufficiency part (unless we impose more structure on the environment such as one-to-one matchings). In Example 2, the second and the third conditions are met. In Example 3, the first and the third conditions are met. In Example, 4, the first and the second conditions are met. As we have discussed, none of the data sets in these examples are rationalizable.*

Remark 3. *We hasten to emphasize that Theorem 1 hinges on the assumption that the schools' priority orderings are known. Without knowledge of the schools' priority orderings, we would not be able to conduct the join and meet operations.*

Before we present the proof of Theorem 1, we briefly discuss the special case of one-to-one matching markets. We note that Theorem 1 can be simplified. This is because the third condition in Theorem 1 is vacuous in one-to-one matching markets. We formally state this observation as a corollary.

Corollary 1. *Suppose that the capacity of each school is one. A data set D is rationalizable if and only if*

(1) $DL \subseteq M$;

(2) the set of students matched and school seats filled is the same for any matching in D.

We now present the proof of Theorem 1. Note that we have established that each of the three conditions is a necessary condition for the rationalizability of a data set in the discussions that lead to Theorem 1. Hence, we shall not repeat these arguments.

Proof of Theorem 1. (The if-part) Suppose that all the conditions hold. We show that that data set D is rationalizable. Our proof is constructive. Step (1) explicitly constructs a class of the students' preferences, and Step (2) verifies that every matching in D is stable given any preferences of the students in this class and the schools' priority orderings.

Step (1). For any two matchings $\mu, \mu' \in DL$ such that $\mu' \geq_S \mu$ and each $i \in I$, if $\mu'(i) \neq \mu(i)$, we define

$$\mu(i) \succ_i \mu'(i). \tag{2}$$

Since the order $\geq_S$ on the set of matchings M is a partial order, we know that $\succ_i$ is a strict partial order.[5] We claim that, for each $i \in I$, $\succ_i$ is complete over $\{\mu(i) : \mu \in DL\}$. To see this, note that for each $i \in I$ and two distinct schools $s', s'' \in \{\mu(i) : \mu \in DL\}$, there exist two matchings $\mu', \mu'' \in DL$ such that $\mu'(i) = s'$ and $\mu''(i) = s''$. Since $\mu' \vee \mu'' \in DL$ by the construction of DL, one of the following cases must occur:

(a) $\mu' \vee \mu'' \geq_S \mu'$ and $(\mu' \vee \mu'')(i) \neq \mu'(i)$;

(b) $\mu' \vee \mu'' \geq_S \mu''$ and $(\mu' \vee \mu'')(i) \neq \mu''(i)$.

Therefore, by the construction in (2), we have either $s' \succ_i s''$ or $s'' \succ_i s'$.

By the Szpilrajn's extension theorem (Szpilrajn (1930), $\succ_i$ can be extended to a strict preference on $S \cup \{i\}$. Thanks to the completeness of $\succ_i$ over $\{\mu(i) : \mu \in DL\}$, we can take a preference profile in the particular class of extensions with the following property:

$$s \succ_i s', \text{ for all } s \in \{\mu(i) : \mu \in DL\} \text{ and all } s' \in S \cup \{i\} \setminus \{\mu(i) : \mu \in DL\}. \tag{3}$$

In words, for each $i \in I$, schools that are never matched to i are in the tail of i's preference list.

Step (2). Next, we verify that D is rationalized by any preference profile that satisfies (2) and (3). Fix such a preference profile $\{\succ_i\}_{i \in I}$. In what follows, we show that every matching in DL is stable given this preference profile of the students and the schools'

[5] A *strict partial order* is a binary relation that is irreflexive, transitive and antisymmetric.

priority orderings, i.e., we show that DL is rationalizable; since $D \subseteq DL$, this implies that D is rationalizable.

Take an arbitrary $\mu \in DL$. We argue that μ is stable by showing that μ is both individually rational and unblocked. First, if μ is not individually rational under $\{\succ_i\}_{i\in I}$, then there exists some i such that $i \succ_i \mu(i)$. By the construction in (2) and (3), $i \succ_i \mu(i)$ only if there exists some $\mu' \in DL$ such that $\mu'(i) = i$. But then, i is matched to some school under μ and i remains unmatched under μ'. We arrive at a contradiction to the second condition. Therefore, μ is individually rational.

Second, if μ is blocked, then there exists a pair (i', s) such that $\mu(i') \neq s$, $s \succ_{i'} \mu(i')$ and $i' \succ_s a$ for some $a \in \mu(s)$. By the construction in (2) and (3), $s \succ_{i'} \mu(i')$ only if there exists $\mu' \in DL$ such that $\mu'(i') = s$ and $\mu \geq_S \mu'$, where the latter implies that $\mu(s) \succ_s \mu'(s)$ since DL is well defined and $\mu(s) \neq \mu'(s)$. Since $\{\mu(s) : \mu \in D\}$ is responsively ordered, we know that $i \succ_s i''$ for all $i \in \mu(s)$ and $i'' \in \mu'(s) - \mu(s)$. Obviously, $i' \in \mu'(s) - \mu(s)$. Therefore, we arrive at a contradiction to $i' \succ_s a$ for some $a \in \mu(s)$. $\square$

The first condition in Theorem 1 states that every matching in DL is a well-defined matching. We now prove a result that simplifies the task of checking whether this condition holds. Theorem 2 below shows that under the second condition and the third condition in Theorem 1, every mapping in DL^+ is a well-defined matching if and only if every mapping in DL^- is a well-defined matching. Formally,

Theorem 2. *For any data set D such that the set of students matched and school seats filled is the same for any matching in D and $\{\mu(s) : \mu \in D\}$ is responsively ordered for all $s \in S$, $DL^+ \subseteq M$ if and only if $DL^- \subseteq M$.*

Proof. Fix a data set D such that the set of students matched and school seats filled is the same for any matching in D and $\{\mu(s) : \mu \in D\}$ is responsively ordered for all $s \in S$. In what follows, we first claim that if DL^+ is well defined, then any school that does not fill its quota at some matching in D is assigned precisely the same set of students at every matching in D, i.e., $s \in \mu(s)$ for some $s \in S$ and some $\mu \in D$ implies that $\mu'(s) = \mu(s)$ for all $\mu' \in D$. To see the claim, suppose there exists $\mu' \in D$ such that $\mu'(s) \neq \mu(s)$. Since DL^+ is well defined, one of the following cases must occur:

(i) $\mu(s) \succ_s \mu'(s)$;
(ii) $\mu'(s) \succ_s \mu(s)$.

Suppose, say, (i) occurs. If $\mu'(s)$ contains more students than $\mu(s)$, then $\mu'(s) - \mu(s)$ contains at least one student. Thus, responsive ordering implies that $s \succ_s i$ for all $i \in \mu'(s) - \mu(s)$.

This is a contradiction to the individual rationality of s under μ'. If $\mu'(s)$ contains less students than $\mu(s)$, then $s \in \mu'(s) - \mu(s)$. Thus, responsive ordering implies that $s \succ_s s$, which is again a contradiction. Symmetric arguments can reach contradictions if (ii) occurs. Therefore, $\mu'(s) = \mu(s)$ for all $\mu' \in D$.

Next, we claim that the set of matched students under $\lambda := \mu \vee \mu'$, denoted by $\lambda(S)$, is the same as the sets under μ and μ', denoted by $\mu(S)$ and $\mu'(S)$, where $\mu(S) = \mu'(S)$ due to condition (2). Obviously, the definition of λ implies that $\lambda(S) \subseteq \mu(S)$. Since any school that does not fill its quota at some matching in D is assigned precisely the same set of students at every matching in D, we know that for each school the number of matched students under λ is the same as that under μ, though the sets of students may intersect across schools. Since λ is matching, we know that those sets of students across schools are mutually disjoint. Therefore, the number of students contained in $\lambda(S)$ is the same as that contained in $\mu(S)$. As a result, $\lambda(S) \subseteq \mu(S)$ implies $\lambda(S) = \mu(S)$.

Now we show that if λ is a matching, then $\nu := \mu \wedge \mu'$ must also be a matching, i.e., (i) $|\nu(i)| = 1$ for every i and $\nu(i) = i$ if $\nu(i) \notin S$; (ii) $|\nu(s)| = q_s$ for every s, and if the number of students in $\nu(s)$, say r, is less than q_s, then $\nu(s)$ contains $q_s - r$ copies of s; and (iii) $\nu(i) = s$ if and only if $i \in \nu(s)$ for all $i \in I$ and all $s \in S$. Since μ and μ' are both matchings, (ii) holds trivially. Since $\nu(i)$, $i \in I$, is defined according to $\nu(s)$, $s \in S$, it then suffices to show that under ν, no student can be assigned to more than one schools. Suppose to the contrary that $i \in \nu(s)$ and $i \in \nu(s')$ for $s \neq s'$. Then one of the following cases must occur because both μ and μ' are matchings:

(a) $\mu(s) \succ_s \mu'(s)$, $i \in \mu'(s)$, $\mu'(s') \succ_{s'} \mu(s')$ and $i \in \mu(s')$;
(b) $\mu'(s) \succ_s \mu(s)$, $i \in \mu(s)$, $\mu(s') \succ_{s'} \mu'(s')$ and $i \in \mu'(s')$.

Note that i can be assigned only to s or s' under λ. Suppose case (a) occurs. Since $i \in \mu'(s)$, we know that $i \notin \mu'(s')$. Similarly, since $i \in \mu(s')$, we know that $i \notin \mu(s)$. Therefore, $i \notin \lambda(s)$ and $i \notin \lambda(s')$, i.e., i is unmatched under λ, a contradiction to $\lambda(S) = \mu(S)$. A symmetric argument can reach the same contradiction if (b) occurs. Hence, ν is a matching.

Induction shows that if $DL^+ \subseteq M$, then $DL^- \subseteq M$. Moreover, the converse is true by symmetric arguments. This completes the proof. $\square$

4 Discussions

4.1 Identification of Possible Preferences

Suppose that a data set D is rationalizable. In this subsection, we discuss the possible preference profiles that rationalize the data set.

For the approach in Section 3.1, recall that we have defined in (1) the set of pairwise relations $R(i)$ for each student i. We say a preference $\succ_i$ over $S \cup \{i\}$ is consistent with $R(i)$ if $(s, s') \in R(i)$ implies $s \succ_i s'$. It is standard to have the following observation (see, for example, Chambers and Echenique (2016)): A preference profile $\{\succ_i\}_{i \in I}$ rationalizes D if and only if $\succ_i$ is consistent with $R(i)$ for all i.

For the approach we studied in Section 3.2, we have defined some pairwise relations in (2). Using (3), we further extend the pairwise relations to a preference profile that rationalizes the data set D. Unlike the extensions of $\{R(i)\}_{i \in I}$, an arbitrary extension of the relations defined in (2) may not rationalize D when it is rationalizable. To see this, we consider a singleton data set $D = \{\mu\}$ such that, say, $\mu(i) = s$. Obviously, D is rationalizable and we must have $s \succ_i i$. However, (2) defines no pairwise relation, which means that the possible extensions include the counterfactual case $i \succ_i s$.

However, we do have the following observation regarding to the relations defined in (2): Suppose D is rationalizable. If for two matchings $\mu, \mu' \in DL$ such that $\mu' \geq_S \mu$ and some student $i \in I$, we have $\mu'(i) \neq \mu(i)$, then $(\mu(i), \mu'(i))$ is in the transitive closure of $R(i)$. This is an implication of the previous observation, since the transitive closure of $R(i)$ is the set of all sure preferences and the relations in (2) must be true when the data set is rationalizable.

4.2 Extreme Stable Matchings

Given a preference-priority profile that satisfies our assumptions, i.e., strictness and responsiveness, we know that the set of stable matchings forms a distributive lattice. Among the stable matchings, the student optimal one and the school optimal one are of particular interest. In our framework, we are also keen to understand how these extreme matchings are related to the revealed preference analysis. Suppose a data set D is rationalizable. On the one hand, knowing that D contains an extreme stable matching will provide us some information of the preferences to be revealed. On the other hand, we may also want to test whether there could be an extreme stable matching in the data set.

Suppose D contains the student optimal stable matching. Then, we can identify a unique candidate of the student optimal stable matching using schools' priorities: there must be a matching $\mu \in D$ such that for any other matching $\mu' \in D$, $\mu'(s) \succsim_s \mu(s)$ for all $s \in S$. Given such a μ, the implication on the unknown student preferences is that $\mu(i) \succsim_i \mu'(i)$ for all $i \in I$ and $\mu' \in D$, where the relation is strict if $\mu(i) \neq \mu'(i)$. The discussion for the school optimal stable matching is symmetric.

To test whether there could be a student optimal stable matching in the data set, it suffices to check if there exists a matching $\mu \in D$ such that for any other matching $\mu' \in D$, $\mu'(s) \succsim_s \mu(s)$ for all $s \in S$. To see this, suppose we have identified such a matching μ. We construct the students' preferences in the same way as in the proof of Theorem 1, except that we need to replace (3) with

$$s \succ_i i \succ_i s', \text{ for all } s \in \{\mu(i) \in S : \mu \in DL\} \text{ and all } s' \in S \setminus \{\mu(i) \in S : \mu \in DL\}. \tag{4}$$

Then, at matching μ, every student is matched with her favorite school. Thus, it is straightforward to verify that μ is the student optimal stable matching under the constructed preference profile, which is also the minimum point of DL. Moreover, the maximum point of DL is the school optimal stable matching.

To test whether there could be a school optimal stable matching in the data set, it suffices to check if there exists a matching $\mu \in D$ such that for any other matching $\mu' \in D$, $\mu(s) \succsim_s \mu'(s)$ for all $s \in S$. To see this, suppose we have identified such a matching μ. We construct the students' preferences in the same way as in the proof of Theorem 1, except that we need to replace (3), again, with (4). Then, at matching μ, every student is matched with the least preferred school among those acceptable ones. Thus, it is straightforward to verify that μ is the student worst (equivalently, school optimal) stable matching under the constructed preference profile, which is also the maximum point of DL. Moreover, the minimum point of DL is the student optimal stable matching.

We summarize these observations in the following proposition.

Proposition 2. *A data set D is rationalizable with $\mu \in D$ being the student optimal stable matching if and only if conditions (1)-(3) hold and for any other matching $\mu' \in D$, $\mu'(s) \succsim_s \mu(s)$ for all $s \in S$. Symmetrically, a data set D is rationalizable with $\mu \in D$ being the school optimal stable matching if and only if conditions (1)-(3) hold and for any other matching $\mu' \in D$, $\mu(s) \succsim_s \mu'(s)$ for all $s \in S$.*

Nevertheless, even if D contains both extreme matchings, DL may not be the set of stable matchings.

Example 5. *Consider a school choice market with four students and two schools whose quotas are both two, i.e.,* $I = \{i_1, i_2, i_3, i_4\}$*,* $S = \{s_1, s_2\}$*, and* $q_{s_1} = q_{s_2} = 2$*. The priority orderings of schools are given as follows:*

$$s_1: \quad i_2, i_4 \succ_{s_1} i_2, i_3 \succ_{s_1} i_2, i_1 \succ_{s_1} i_4, i_3 \succ_{s_1} i_4, i_1 \succ_{s_1} i_3, i_1 \succ_{s_1} i_2 \succ_{s_1} i_4 \succ_{s_1} i_3 \succ_{s_1} i_1 \succ_{s_1} s_2$$
$$s_2: \quad i_1, i_3 \succ_{s_2} i_1, i_2 \succ_{s_2} i_1, i_4 \succ_{s_2} i_3, i_2 \succ_{s_2} i_3, i_4 \succ_{s_2} i_2, i_4 \succ_{s_2} i_1 \succ_{s_2} i_3 \succ_{s_2} i_2 \succ_{s_2} i_4 \succ_{s_2} s_1.$$

The two orderings are obviously responsive. The data set consists of the following two observations $D = \{\mu, \mu'\}$*, where*

$$\mu = \begin{matrix} i_2, i_4 & i_1, i_3 \\ s_1 & s_2 \end{matrix} \qquad \textit{and} \qquad \mu' = \begin{matrix} i_1, i_3 & i_2, i_4 \\ s_1 & s_2 \end{matrix}.$$

It is easy to see that $DL = D = \{\mu, \mu'\}$ *and that the data set is rationalizable. More precisely, the revealed student preferences from (2) are:*

$$\begin{matrix} i_1: & s_1 \succ_{i_1} s_2 & \qquad & i_2: & s_2 \succ_{i_2} s_1 \\ i_3: & s_1 \succ_{i_3} s_2 & \qquad & i_4: & s_2 \succ_{i_4} s_1 \end{matrix}$$

However, the set of stable matchings under the revealed preferences and the school priorities also includes, say,

$$\mu'' = \begin{matrix} i_3, i_4 & i_1, i_2 \\ s_1 & s_2 \end{matrix}.$$

4.3 Assumptions on Preferences

Our model makes two assumptions over the students' preferences and the schools' priority orderings, namely, strictness and responsiveness. More precisely, we assume that the students' preferences over schools are strict, and the schools' priority orderings over individual students are strict. We also assume that the schools' priority orderings over sets of students are responsive. In this subsection, we investigate the relaxation of these assumptions.

When the students' preferences or the schools' priority orderings over individual students are not strict, we can still test the rationalizability of a data set using the first approach in Section 3.1. The only difference is in the interpretation of $R(i)$. Precisely, in this case $(a, a') \in R(i)$ means that a is revealed to be weakly preferred to a' by student i. In contrast, the approach that we developed in Section 3.2 is not applicable any more when we drop the strictness assumption. Particularly, Theorem 1 fails because the set of stable matchings does not have any of the three properties (see Roth and Sotomayor (1990, pp. 48-50 and pp. 162-163) for examples).

When preferences (priorities over individuals) are strict but not necessarily responsive, again, we can still test the rationality of a data set as in Section 3.1, except that the collection of schools that are revealed to be available to student i under matching μ_k is now defined as

$$A(i, \mu_k) := \{s \in S : s \neq \mu_k(i) \text{ and } \mu_k(s) \cup \{i\} \succ_s \mu_k(s)\}.$$

It is known that stable matchings exist under more general school priorities such as substitutable priorities (see Hatfield and Milgrom (2005)) and bilateral substitutable priorities (see Hatfield and Kojima (2008) and Hatfield and Kojima (2010)). However, the lattice structure of the set of stable matchings may or may not be preserved. For example, it is not preserved under bilateral substitutable priorities but is preserved under the substitutable priorities. When the lattice structure is not preserved, our analysis in Section 3.2 cannot be applied. When it is preserved as in Martínez et al. (2001) and Hatfield and Milgrom (2005), the lattice is with respect to the students' (doctors' or workers' in their papers) preferences, which is unknown in our context such that we cannot apply the join and meet operators.[6]

4.4 Supply and Demand

Azevedo and Leshno (2016) develop a framework to apply the supply and demand analysis to matching markets. In our paper, since schools' priority orderings are known, each student's budget set can be recovered from the observed matchings in the data set. To connect with the classic framework of Afriat (1967), for each school $s \in S$, let $\theta_s^k \in R^N$ denote the characteristic of school s at period k. Each dimension of θ_s^k is an attribute of the school. For example, the first dimension could represent the reputation of the school

[6]In the context of matching with contracts, i.e., Hatfield and Milgrom (2005), the lattice structure is with respect to the doctors in the following sense: Given a stable allocation, we need to recover the opportunity sets of doctors and hospitals in order to pin down a point in the lattice, where the recovery uses the doctors preferences. Technically, one could study the case where the lattice structure is preserved, the students' preferences are known, and the schools' priorities are to be revealed. Since Examples 3 and 4 show the insufficiency of $DL \subseteq M$, one needs more properties of stable matchings such as (2) and (3) in Theorem 1 to guarantee the rationalizability of a data set. However, for example, the rural hospital property is preserved only under additional assumptions, such as the law of aggregate demand in Hatfield and Milgrom (2005); and it is unclear how responsive ordering may be preserved. We leave such an extension as an open question.

and the second dimension could represent the average salary of students after graduation.

The preferences of students could be defined over R^N. We could restrict our attention to strictly monotone preferences, i.e. $\theta > \theta'$ implies that θ is strictly preferred to θ'. We adopt the framework of Nishimura et al. (2017). Consider a preorder $\trianglerighteq$ defined over R^N such that $\theta \trianglerighteq \theta'$ if and only if $\theta \geq \theta'$. Let $\triangleright$ denote the asymmetric part of the preorder. We seek to identify a preference relation $\succsim^i$ for each student i such that $\succsim^i$ extends the preorder $\trianglerighteq$[7] and strictly rationalizes[8] the matching outcome.

Formally, fix the schools' preferences and capacities $\{q_s, \succ_s\}_{s\in S}$. For each observation $k \in K$, let $\Theta^k = \{\theta^k_s\}_{s\in S} \cup \{\mathbf{0}\} \subset R^N$ denote the set of schools' characteristics at period k where $\mathbf{0} = (0, ..., 0)$ is the outside option of not matching. We could recover the budget set of agent i at period k as $B^k_i \subset \Theta^k$. With the observed matching μ_k, we could recover the choice function of agent i as c_i such that $c_i(B^k_i) = \mu_k(i)$. Denote by $\mathcal{A}_i$ as the set of budgets of agent i.

For each agent i, we have established her choice environment $((R^N, \trianglerighteq), \mathcal{A}_i)$ and the attached choice function c_i. Here, we want to recover the preference of student i over R^N such that the preference strictly rationalizes the choice of the student while maintaining the property of monotonicity, i.e., extending the preorder $\trianglerighteq$. Nishimura et al. (2017) show how to perform the extension. In particular, when the underlying preorder is continuous[9], they provide a sufficient and necessary condition for extending the preorder to a continuous preference while strictly rationalizing the observed choice function.

4.5 Multiple observations

In this subsection, we discuss several settings in which one has multiple observations of (stable) matchings. We first consider a school choice setting in which we assume that individuals with the same characteristics are identical and have identical preferences. We then analyze a graduate rotational program in large corporations. Finally, we study an evolving labor market with retirements and new entries.

School choice setting. In a school choice setting, while the schools remain active for multiple periods, the students are different in each period. Thus, the matchings are not

[7] A complete pre-order $\succsim$ is said to extend $\trianglerighteq$ if $\theta \trianglerighteq \theta'$ implies $\theta \succsim \theta'$ and $\theta \triangleright \theta'$ implies $\theta \succ \theta'$ where $\succ$ is the asymmetric part of $\succsim$.

[8] By strict rationalization, we mean that the student strictly prefers the matched school within the budget set, which is consistent with the whole framework of our paper.

[9] Consider a topological space X. A binary relation $R \subset X \times X$ is said to be continuous if R is a closed subset of $X \times X$.

among the same set of agents. As in Echenique et al. (2013), we assume that students with the same characteristics are identical and have identical preferences. In this case, we have multiple observations of matchings that fit our framework. While this is a strong assumption, without it, the theory has no testable implications in the following sense: any data set such that every matching in the data set is individually rational for all school could be trivially rationalizable.

Graduate rotational program. In many large corporations, new employees rotate among different divisions for multiple rounds. In each round, bilateral selection produces a new matching. The pool of agents is the same in all rounds.

Our main result is not a complete characterization of the rationalizability of a data set in this setting. This is because the matchings produced from the Graduate Rotational Program are history dependent, i.e., a particular match appears at most once. If employee i is matched with division s in a particular round, employee i will not be matched with division s in other rounds, even if i ranks s highest and s ranks i highest.

The sufficiency part of our analysis still has a bite. If a data set is rationalizable when we ignore the constraint that a particular match appears at most once, then it is rationalizable with the constraint. The necessity part does not apply, as illustrated by the Example 6 below.

Example 6 (Example 1 revisited)**.** *Consider the matchings between employees in $I = \{i_1, i_2, i_3\}$ and divisions in $S = \{s_1, s_2, s_3\}$. The capacity of each division is one. The divisions' priority orderings over individual students are given as follows:*

$$s_1 : i_2 \succ_{s_1} i_3 \succ_{s_1} i_1 \succ_{s_1} s_1$$

$$s_2 : i_3 \succ_{s_2} i_1 \succ_{s_2} i_2 \succ_{s_2} s_2$$

$$s_3 : i_1 \succ_{s_3} i_2 \succ_{s_3} i_3 \succ_{s_3} s_3$$

The data set consists of the following two observations $D = \{\mu, \mu'\}$, where

$$\mu = \begin{matrix} i_1 & i_2 & i_3 \\ s_1 & s_3 & s_2 \end{matrix} \qquad \text{and} \qquad \mu' = \begin{matrix} i_1 & i_2 & i_3 \\ s_3 & s_2 & s_1 \end{matrix}.$$

We have shown in Example 1 and Example 2 that the data set is not rationalizable when we ignore the constraint that a particular match appears at most once. We argue that if μ happens earlier than μ', then this data set is rationalizable with the constraint. In particular, *let the employees' preferences be as follows:*

$$i_1 : s_1 \succ_{i_1} s_3 \succ_{i_1} s_2 \succ_{i_1} i_1$$

$$i_2 : s_3 \succ_{i_2} s_2 \succ_{i_2} s_1 \succ_{i_2} i_2$$

$$i_3 : s_2 \succ_{i_3} s_1 \succ_{i_3} s_3 \succ_{i_3} i_3$$

Then, μ is stable. Moreover, μ' is stable because s_1 has been matched with s_1, i_2 has been matched with s_3, and i_3 has been matched with s_2 in previous rounds.

Evolving labor market. In an evolving labor market (see, for example, Blum et al. (1997)), we have observations of different matchings at different stages. Although the set of agents evolves over time due to retirements and new entries, our analysis can be used to refute the rationalizability of a data set.

We consider multiple matchings, where the sets of agents in different matchings are not necessarily the same. We note that if a matching is stable, then any sub-matching, which is a restriction of the original matching on a subset of agents such that no match is broken, is stable. Therefore, by taking a subset of the data set and restricting attention to the set of common agents such that they are matched only to agents in the set under all data points, we have a data set that fits our framework. Our theory, which can be applied to an arbitrary subset of the data set, provides a necessary condition for the original data set to be rationalizable.

Example 7 shows that our conditions are not sufficient.

Example 7. *Consider the following evolving labor market with two periods $t = 1, 2$. The capacity of each firm in each period is one. The set of agents in each period is reflected in the matchings:*

$$\mu_1 = \begin{matrix} i_1 & i_3 \\ s_1 & s_2 \end{matrix} \qquad \text{and} \qquad \mu_2 = \begin{matrix} i_2 & i_3 \\ s_2 & s_1 \end{matrix}.$$

The schools' priority orderings are given as follows:

$$s_1 : i_2 \succ_{s_1} i_3 \succ_{s_1} i_1 \succ_{s_1} s_1$$

$$s_2 : i_3 \succ_{s_2} i_1 \succ_{s_2} i_2 \succ_{s_2} s_2.$$

Since there exists no common sub-matching where the matched agents are the same, our conditions on common sub-matchings are trivially satisfied. However, the data set is not *rationalizable,since the stability of μ_0 requires that $s_2 \succ_{i_3} s_1$ and the stability of μ_1 requires that $s_1 \succ_{i_3} i_2$.*

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